

Colorful SU(2) center vortices in the continuum and on the lattice

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The spherical vortex as introduced in [Phys. Rev. **D77**, 014515 (2008)] is generalized. A continuum form of the spherical vortex is derived and investigated in detail. The discrepancy between the gluonic lattice topological charge and the index of the lattice Dirac operator described in previous papers is identified as a discretization effect. The importance of the investigations for Monte Carlo configurations is discussed.

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I. INTRODUCTION

The distribution of topological charge density is closely linked to the phenomena of the axial anomaly and spontaneous chiral symmetry breaking. There are strong hints that the QCD vacuum is dominated by center vortices [1–6]. Center vortices can explain color confinement [7–10] and seem to be also of paramount importance for spontaneous chiral symmetry breaking [11–15]. Center vortices can contribute to the topological charge density through intersection and writhing points [11], but also through their color structure. The prototype of the later contribution in SU(2) gauge theory is the spherical vortex introduced in previous articles of our group [16–18]. In this paper, a generalization of the original spherical vortex is constructed. Subsequently, the generalized spherical vortex is investigated in detail on the lattice and in the continuum.

In section II, we start with a description of the original spherical vortex on the lattice as introduced in [16]. From this lattice gauge configuration, we then derive the corresponding continuum object. With the continuum form, a more general spherical vortex can be described. The action and topological charge density of such generalized spherical vortices in the continuum are discussed in section III and IV. We show that the large action of the original spherical vortex encountered in previous investigations can be significantly reduced by spreading the vortex over several time slices. In section V, a generalized spherical vortex on the lattice is derived from the continuum form. Subsequently its gluonic and fermionic properties are investigated. In particular, we show that the discrepancy between the gluonic lattice topological charge and the index of the lattice Dirac operator seen for the original spherical vortex is simply a discretization effect. We conclude with remarks on the contributions of colorful center vortices to topological charge and chiral symmetry breaking in Monte Carlo configurations.

II. THE ORIGINAL SPHERICAL VORTEX

The original spherical vortex on the lattice as introduced in [16] is given by the lattice links [19]

$$\begin{aligned} U_i(x) &= \mathbb{1} \\ U_4(x) &= \begin{cases} \cos[\alpha(|\vec{r} - \vec{r}_0|)] \mathbb{1} - i \vec{e}_r \cdot \vec{\sigma} \sin[\alpha(|\vec{r} - \vec{r}_0|)] & \text{for } t = 1 \\ \mathbb{1} & \text{else} \end{cases} \end{aligned} \quad (1)$$

Here $x = \{x_1, x_2, x_3, t\}$ stands for the lattice site and \vec{r} denotes the spatial components only. The unit vector pointing from the vortex midpoint \vec{r}_0 to the spatial lattice site \vec{r} is denoted by

$$\vec{e}_r = \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|}.$$

$U_\mu(x)$ is the link connecting the lattice site x with the site $x + \hat{\mu}$, where $\hat{\mu}$ is the unit vector pointing in the μ direction. The gauge links on the lattice represent the adjoint gauge transporters and, therefore, transform like $U_\mu(x) \rightarrow \Omega(x + \hat{\mu}) U_\mu(x) \Omega(x)^\dagger$ under a gauge transformation $\Omega(x)$. Indices denoted by Latin letters run from 1 to 3 (spatial indices), while indices denoted by Greek letters run from 1 to 4 (spatial plus temporal indices). The function $\alpha(r)$ rises linearly from $-\pi$ to 0, *i.e.*,

$$\alpha(r) = -\pi \cdot \begin{cases} 1 & \text{for } r < R - \frac{d}{2} \\ \frac{R + \frac{d}{2} - r}{\frac{d}{2}} & \text{for } R - \frac{d}{2} \leq r \leq R + \frac{d}{2} \\ 0 & \text{for } r > R + \frac{d}{2} \end{cases} \quad (2)$$

Note that the spherical vortex configuration possesses only non-trivial temporal links U_4 . All spatial links U_i are trivial and therefore the gluonic lattice topological charge $Q = 0$. On the other hand the index of the Dirac operator $\text{ind}[D] = n_- - n_+$, defined as the difference between the number of left- and right-handed chiral zero modes, is always 1. This gives a discrepancy between the index and the gluonic topological charge. Note that no violation of the index theorem $\text{ind}[D] = Q$ occurs as the theorem is only stated for the continuum limit. When cooling, the gluonic topological charge approaches 1 and the discrepancy $\text{ind}[D] \neq Q$ gets resolved [16–18].

Naively, one would perform the continuum limit of the spherical vortex by simply splitting the temporal links and leaving the spatial links trivial. This would give gauge configurations of the form

$$\begin{aligned} U_i(x) &= \mathbb{1}, \\ U_4(x) &= \cos[w(t) \alpha(|\vec{r} - \vec{r}_0|)] \mathbb{1} - i \vec{e}_r \cdot \vec{\sigma} \sin[w(t) \alpha(|\vec{r} - \vec{r}_0|)] \quad \text{with} \quad \sum_t w(t) = 1, \end{aligned} \quad (3)$$

where $w(t)$ determines how the temporal links are split. As one can easily see, a singularity occurs at the vortex midpoint when $0 \neq w(t) \neq 1$. Moreover, the gluonic topological charge Q clearly remains 0 independent of $w(t)$ which doesn't agree with the cooling history of the original vertex, Eq. (1). Therefore, this simple procedure seems not to be suitable for performing the continuum limit of the lattice spherical vortex.

In order to find a better continuum limit and to gain understanding of the spherical vortex, one has to apply a gauge transformation to the link configuration given in Eq. (1). With the lattice gauge transformation

$$\Omega(x) = \begin{cases} g(\vec{r}) & \text{for } 1 < t \leq t_g \\ \mathbb{1} & \text{else} \end{cases},$$

where

$$g(\vec{r}) = \cos[\alpha(|\vec{r} - \vec{r}_0|)] \mathbb{1} + i \vec{e}_r \cdot \vec{\sigma} \sin[\alpha(|\vec{r} - \vec{r}_0|)], \quad (4)$$

the spherical vortex becomes

$$\begin{aligned} U_i(x) &= \begin{cases} g(\vec{r} + \hat{i}) g(\vec{r})^\dagger & \text{for } 1 < t \leq t_g \\ \mathbb{1} & \text{else} \end{cases}, \\ U_4(x) &= \begin{cases} g(\vec{r})^\dagger & \text{for } t = t_g \\ \mathbb{1} & \text{else} \end{cases}. \end{aligned} \quad (5)$$

From this it becomes clear that the spherical vortex represents a transition (in the temporal direction) between two pure gauge fields. The transition occurs between $t = 1$ and $t = 2$. No transition occurs between $t = t_g$ and $t = t_g + 1$ since all plaquettes there are trivial. For $t \leq 1$ the pure gauge field is trivial, *i.e.*, the winding number $n_w = 0$. For $t > 1$ the pure gauge field is generated by the hedgehog gauge transformation, Eq. (4), which has $n_w = 1$ [20]. This means that in the continuum limit, the spherical vortex (representing a transition between a vacuum with winding number 0 and 1) has topological charge $Q = 1$ [20]. It can be regarded as a squeezed instanton with center vortex structure.

Assuming an infinitely big temporal extent of the lattice and taking $t_g \rightarrow \infty$, the continuum field corresponding to Eq. (5) can be written as

$$\mathcal{A}_\mu = f(t) i (\partial_\mu g) g^\dagger, \quad (6)$$

where g is given in Eq. (4) and $f(t)$, determining the transition in temporal direction t , changes from 0 to 1 within one lattice unit. Clearly, one could also use a function $f(t)$ that changes more slowly. In this case, the corresponding lattice object cannot be represented by configurations of the form of Eq. (1) anymore. We refer to the construction with a smoother function $f(t)$ as a generalized spherical vortex. One construction of such a generalized spherical vortex on the lattice will be investigated in section V. Before doing so, we discuss the action and topological charge density of the continuum gauge field, given in Eq. (6) in the next two sections.

III. ACTION OF THE SPHERICAL VORTEX IN THE CONTINUUM

In this section, we calculate the action for the generalized spherical vortex in the continuum given in Eq. (6). For simplicity, we set the vortex midpoint \vec{r}_0 to 0 in this and the next section. Moreover, we use the notation $|\vec{r}| = r$.

To explicitly evaluate the action S for the \mathcal{A}_μ given in Eq. (6), we have to evaluate $i (\partial_\mu g) g^\dagger$ where g is given in Eq. (4). To do so, we first have a look at $i (\partial_\mu g) g^\dagger$ for a general gauge transformation

$$g = q_0 \sigma_0 + i \vec{q} \cdot \vec{\sigma},$$

where q_0 and \vec{q} are constraint by $q_0^2 + |\vec{q}|^2 = 1$. Using the identity $\sigma^a \sigma^b = \delta^{ab} \mathbb{1} + i \epsilon^{abc} \sigma^c$ and

$$q_0 \partial_\mu q_0 + q_k \partial_\mu q_k = \frac{1}{2} \partial_\mu (q_0 q_0 + q_k q_k) = \frac{1}{2} \partial_\mu (1) = 0,$$

one gets, after a few lines of calculation,

$$i (\partial_\mu g) g^\dagger = \vec{\sigma} \cdot [(\partial_\mu q_0) \vec{q} - q_0 (\partial_\mu \vec{q}) + \vec{q} \times (\partial_\mu \vec{q})]. \quad (7)$$

For the g given in Eq. (4) we have $q_0 = \cos(\alpha(r))$ and $\vec{q} = \vec{e}_r \sin(\alpha(r))$. Inserting this into Eq. (7) and multiplying with $f(t)$ gives $(\mathcal{A}_\mu = \frac{\sigma_a}{2} A_\mu^a)$

$$\begin{aligned} A_i^a &= f(t) \cdot 2 \cdot \left(\frac{x_i x_a}{r^3} \sin(\alpha(r)) \cos(\alpha(r)) + \epsilon_{iak} x_k \frac{1}{r^2} \sin^2(\alpha(r)) \right. \\ &\quad \left. - \delta_{ia} \frac{1}{r} \sin(\alpha(r)) \cos(\alpha(r)) - \frac{x_i x_a}{r^2} \alpha'(r) \right), \\ A_4^a &= 0. \end{aligned} \quad (8)$$

In the following, we use the notation

$$A_\mu^a = f(t) A_{\mu+}^a. \quad (9)$$

The field strength tensor is given by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - \epsilon^{abc} A_\mu^b A_\nu^c.$$

The different elements of the field strength tensor can be simplified considerably for our particular gauge field configuration. Let us first have a look at the elements of the form F_{4i}^a . Using the fact that the gauge field has no temporal components, *i.e.*, $A_4^a = 0$, we can simplify these elements to

$$F_{4i}^a = \partial_4 A_i^a. \quad (10)$$

The F_{ij}^a can be simplified too. With the notation (9) we get

$$\begin{aligned} F_{ij}^a &= f(t) \partial_i A_{j+}^a - f(t) \partial_j A_{i+}^a - f(t)^2 \epsilon^{abc} A_{i+}^b A_{j+}^c \\ &= f(t) (\partial_i A_{j+}^a - \partial_j A_{i+}^a - \epsilon^{abc} A_{i+}^b A_{j+}^c) + (f(t) - f(t)^2) \epsilon^{abc} A_{i+}^b A_{j+}^c \\ &= f(t) (1 - f(t)) \epsilon^{abc} A_{i+}^b A_{j+}^c. \end{aligned} \quad (11)$$

Using these results, we can now evaluate the action. Let us first evaluate $\text{tr}_C [\mathcal{F}_{4i} \mathcal{F}_{4i}]$. With (10) and (9) we get

$$\text{tr}_C [\mathcal{F}_{4i} \mathcal{F}_{4i}] = \frac{1}{2} F_{4i}^a F_{4i}^a = \frac{1}{2} \left(\frac{d}{dt} f(t) \right)^2 A_{i+}^a A_{i+}^a.$$

Inserting the explicit form of A_{i+}^a one can evaluate

$$A_{i+}^a A_{i+}^a = 4 \left(\frac{2}{r^2} \sin(\alpha(r))^2 + \alpha'(r)^2 \right).$$

Similarly we get

$$\text{tr}_C [\mathcal{F}_{ij} \mathcal{F}_{ij}] = \frac{1}{2} F_{ij}^a F_{ij}^a = f(t)^2 (1 - f(t))^2 \epsilon^{abc} A_{i+}^b A_{j+}^c \epsilon^{ade} A_{i+}^d A_{j+}^e,$$

where Eq. (11) has been used in the last step. With the help of a computer algebra program, one can evaluate

$$\epsilon^{abc} A_{i+}^b A_{j+}^c \epsilon^{ade} A_{i+}^d A_{j+}^e = \frac{32}{r^4} \sin(\alpha(r))^2 \left(\sin(\alpha(r))^2 + 2r^2 \alpha'(r)^2 \right).$$

Combining these identities and switching to polar coordinates gives

$$\begin{aligned} S &= \frac{1}{2g^2} \int d^4x \text{tr}_C [\mathcal{F}_{\mu\nu} \mathcal{F}_{\mu\nu}] = \frac{1}{2g^2} \int d^4x (\text{tr}_C [\mathcal{F}_{ij} \mathcal{F}_{ij}] + 2 \text{tr}_C [\mathcal{F}_{4i} \mathcal{F}_{4i}]) \\ &= \frac{1}{2g^2} \left(\int dt f(t)^2 (1 - f(t))^2 \int dr \frac{64\pi}{r^2} \sin(\alpha(r))^2 \left(\sin(\alpha(r))^2 + 2r^2 \alpha'(r)^2 \right) \right. \\ &\quad \left. + \int dt \left(\frac{d}{dt} f(t) \right)^2 \int dr 16\pi \left(2 \sin(\alpha(r))^2 + r^2 \alpha'(r)^2 \right) \right). \end{aligned} \quad (12)$$

So far, the calculation has been done for a general $f(t)$. Now we will choose $f(t)$ as the piecewise linear function

$$f_{\Delta t}(t) = \begin{cases} 0 & \text{for } t < 1 \\ \frac{t-1}{\Delta t} & \text{for } 1 \leq t \leq 1 + \Delta t, \\ 1 & \text{for } t > 1 + \Delta t \end{cases} \quad (13)$$

where Δt stands for the duration of the transition. Inserting $f_{\Delta t}(t)$ into Eq. (12) gives

$$S = \frac{1}{2g^2} \left(\Delta t \cdot \int dr \frac{32\pi}{15r^2} \sin(\alpha(r))^2 \left(\sin(\alpha(r))^2 + 2r^2 \alpha'(r)^2 \right) + \frac{1}{\Delta t} \cdot \int dr 16\pi \left(2 \sin(\alpha(r))^2 + r^2 \alpha'(r)^2 \right) \right). \quad (14)$$

One can see from Eq. (14), that as long as none of the spatial integrations gives zero, the action diverges for both $\Delta t \rightarrow \infty$ and $\Delta t \rightarrow 0$. Note that the first term in Eq. (14) represents the magnetic and the second term the electric contributions to the action. This means that for $\Delta t \rightarrow 0$, the action is purely electric, and for $\Delta t \rightarrow \infty$, it is purely magnetic. In between, there is a minimum where the electric and magnetic contributions are equal.

Performing the spatial integration is a difficult task, which was done with a computer algebra program. The result for a general ratio of d/R is lengthy and therefore not explicitly given here. However, let us state some properties of the result. If we minimize the action for a given R and d with respect to Δt , we get an expression that depends only on the ratio d/R . This expression falls monotonically with d/R in the allowed range $0 \leq d/R \leq 2$. It diverges for $d/R \rightarrow 0$ and approaches its minimum for $d/R \rightarrow 2$. The absolute minimum of the action (minimized with respect to Δt , R and d) is given by $S_{min} = 1.667 S_{Inst}$ ($R = d/2 \approx 0.305 \Delta t$). Here we denoted the action of one instanton by $S_{Inst} = 1/(2g^2) 16\pi^2$. This action serves as a lower bound for objects with topological charge $|Q| = 1$. Fixing the ratio d/R gives the action as a function of R and Δt . For $d/R = 1$ we get

$$\frac{S(\Delta t)}{S_{Inst}} = \frac{0.4358}{R} \cdot \Delta t + \frac{3.722}{\Delta t} R. \quad (15)$$

IV. TOPOLOGICAL CHARGE DENSITY OF THE SPHERICAL VORTEX IN THE CONTINUUM

In this section, we calculate the topological charge density $q(x)$ of field configurations of the form of Eq. (6). For now, the calculation is done for a general gauge transformation g and a general $f(t)$ with the restriction, that they have to be chosen in such a way, that the resulting fields are free of singularities. Moreover, we will assume that g is independent of the temporal coordinate t . In the following, we again use the shorthand notation $\mathcal{A}_{i+} = i(\partial_i g)g^\dagger$.

As is well known [20], $q(x)$ can be calculated as the full derivative

$$q(x) = \partial_\mu K_\mu(x), \quad \text{with} \quad K_\mu = \frac{1}{16\pi^2} \epsilon_{\mu\alpha\beta\gamma} \left(A_\alpha^a \partial_\beta A_\gamma^a - \frac{1}{3} \epsilon^{abc} A_\alpha^a A_\beta^b A_\gamma^c \right).$$

Let us now split $\partial_\mu K_\mu$ into a spatial ($\partial_i K_i$) and a temporal part ($\partial_4 K_4$). First, we treat the spatial part. With the assumption that g is independent of the temporal coordinate, one can easily see that the gauge field (6) has no temporal component, i.e. $\mathcal{A}_4 = 0$. Therefore, K_i evaluates to zero:

$$\begin{aligned} K_i &= \frac{1}{16\pi^2} \epsilon_{i\alpha\beta\gamma} \left(A_\alpha^a \partial_\beta A_\gamma^a - \frac{1}{3} \epsilon^{abc} A_\alpha^a A_\beta^b A_\gamma^c \right) = \frac{1}{16\pi^2} \epsilon_{i\alpha 4 \gamma} A_\alpha^a \partial_4 A_\gamma^a \\ &= -\frac{1}{16\pi^2} \epsilon_{ijk} A_j^a \partial_4 A_k^a = -\frac{1}{16\pi^2} \epsilon_{ijk} A_{j+}^a A_{k+}^a f(t) f'(t) = 0. \end{aligned}$$

This means, that we can identify the topological charge density with the temporal derivative of K_4 , i.e.,

$$q(x) = \partial_4 K_4.$$

From the definition of K_μ given above, we see that we can write K_4 as

$$K_4 = -\frac{1}{16\pi^2} \left(f(t)^2 \epsilon_{ijk} A_{i+}^a \partial_j A_{k+}^a - f(t)^3 \frac{1}{3} \epsilon_{ijk} \epsilon^{abc} A_{i+}^a A_{j+}^b A_{k+}^c \right),$$

where we have used $\epsilon_{4ijk} = -\epsilon_{ijk}$. Using the fact that the field strength vanishes for pure gauge fields, *i.e.*, $F_{\mu\nu}^a = \partial_\mu A_{\nu+}^a - \partial_\nu A_{\mu+}^a - \epsilon^{abc} A_{\mu+}^b A_{\nu+}^c = 0$, one can rewrite the term $\epsilon_{ijk} A_{i+}^a \partial_j A_{k+}^a$ as $\frac{1}{2} \epsilon_{ijk} \epsilon^{abc} A_{i+}^a A_{j+}^b A_{k+}^c$. Using this result, we get

$$K_4 = -\frac{1}{16\pi^2} \left(\left(\frac{1}{2} f(t)^2 - \frac{1}{3} f(t)^3 \right) \epsilon_{ijk} \epsilon^{abc} A_{i+}^a A_{j+}^b A_{k+}^c \right). \quad (16)$$

The topological charge density is simply the temporal derivative of this expression. Note that $\epsilon_{ijk} \epsilon^{abc} A_{i+}^a A_{j+}^b A_{k+}^c$ is proportional to the winding number density of the gauge transformation. This means that the spatial dependence of the topological charge density is given by the winding number density and the temporal dependence is given by $(1 - f(t)) f(t) f'(t)$. With the g defined in Eq. (4), the $\alpha(r)$ defined in Eq. (2) and the $f(t)$ defined in Eq. (13) the topological charge density evaluates to

$$q(r, t) = \frac{3}{d\pi r^2} \cos^2 \left(\frac{\pi(r - R)}{d} \right) \left(\frac{1}{4\Delta t} - \frac{t^2}{\Delta t^3} \right) \cdot \begin{cases} 1 & \text{for } 1 \leq t \leq 1 + \Delta t \text{ and } R - \frac{d}{2} \leq r \leq R + \frac{d}{2} \\ 0 & \text{else} \end{cases}. \quad (17)$$

As can easily be checked, integrating this expression ($r^2 dr$ and dt) yields $Q = 1$.

V. THE GENERALIZED SPHERICAL VORTEX ON THE LATTICE

We now put the generalized continuum spherical vortex of Eq. (6) onto the lattice with periodic boundary conditions. Clearly, the field as given in Eq. (6) does not fulfill periodic boundary conditions in the temporal direction. \mathcal{A}_μ vanishes for $t \rightarrow -\infty$ (and also for $r \rightarrow \infty$) but not for $t \rightarrow \infty$. If we want to get a vanishing gauge field \mathcal{A}_μ for $t \rightarrow \infty$ by gauge transforming the field of Eq. (6), we have to find a gauge transformation that equals $\mathbb{1}$ for $t \rightarrow -\infty$ and g^\dagger for $t \rightarrow \infty$. As can be shown easily by continuity arguments, there is no continuous gauge transformation that fulfills that criteria for the g given in Eq. (4). However, there is still the possibility to put the field onto a lattice of infinite size. Then, one can transform the links for $t \rightarrow \infty$ to unity and subsequently close the lattice by periodic boundary conditions. Performing this procedure for the continuum field of Eq. (6) with $f(t)$ given in Eq. (13) yields

$$U_i(x) = \begin{cases} \left(g \left(\vec{r} + \hat{i} \right) g(\vec{r})^\dagger \right)^{(t-1)/\Delta t} & \text{for } 1 < t < 1 + \Delta t \\ g \left(\vec{r} + \hat{i} \right) g(\vec{r})^\dagger & \text{for } 1 + \Delta t \leq t \leq t_g \\ \mathbb{1} & \text{else} \end{cases}, \quad (18)$$

$$U_4(x) = \begin{cases} g(\vec{r})^\dagger & \text{for } t = t_g \\ \mathbb{1} & \text{else} \end{cases}.$$

The functions $g(\vec{r})$ and $\alpha(r)$ are again given by Eqs. (4) and (2).

Let us now have a look at the gauge action and topological charge of this generalized spherical vortex on the lattice. The gauge action as function of the temporal extent Δt is plotted in Fig. 1a). Note, that the action of the lattice object matches the action of the underlying continuum object pretty well. However, it systematically underestimates the continuum value. In Fig. 1b) we show the topological charge density as function of Δt for three values of $R = d$. As one can see, the discrepancy between the topological charge Q on the lattice and in the continuum (which is always 1) is dramatic for small temporal extent Δt of the spherical vortex. For $\Delta t = 1$ the topological charge of the continuum object is not recognized. However, one still gets a zeromode for the corresponding lattice Dirac operator, *i.e.*, the fermions still see the topological charge of the underlying continuum object. This is not unexpected as the vacuum to vacuum transition is still present in the lattice object. To be more precise, the lattice samples the field before and after the transition. The transition itself falls between two time-slices. As discussed in section IV, the transition carries the topological charge. Therefore, by missing the transition, one also misses the topological charge. From Fig. 1 we see, that also for very big Δt (slow vacuum to vacuum transitions), the lattice topological charge doesn't quite approach one. However, increasing R , keeping $R = d$ in the example shown, increases Q towards one. Since R is the only spacial scale, $1/R$ acts as the spacial lattice spacing, showing that the deviations from $Q = 1$ at large Δt ($\Delta t \gtrsim 10$) are discretization effects in the spatial directions.

To conclude this section we briefly discuss how the fermionic results depend on the temporal extent of the vortex. For the fermion fields we use antiperiodic boundary conditions in the temporal direction and periodic boundary conditions in the spatial directions. With these boundary conditions, the Dirac operator possesses always exactly one zeromode for gauge configurations of the form of Eq. (18), independent of the values of the parameters. This

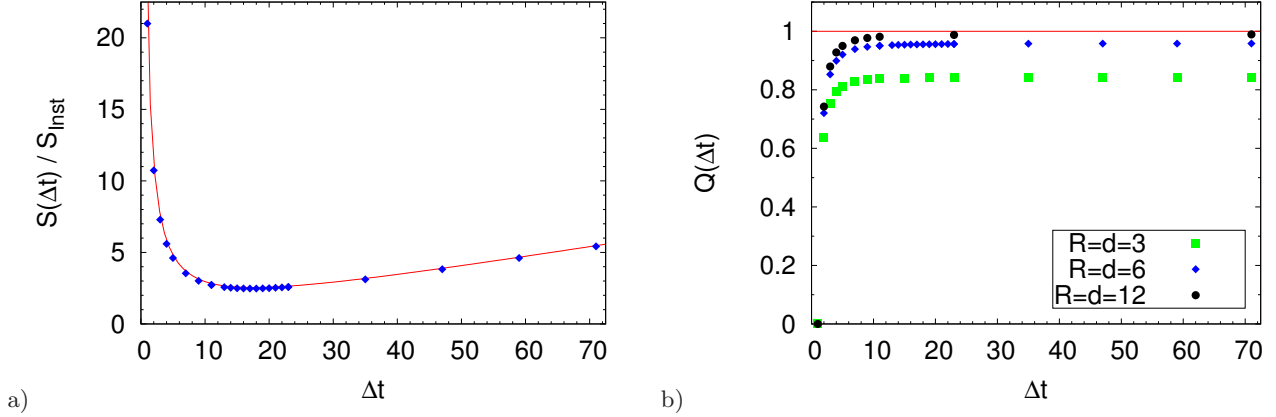


FIG. 1. a) Comparison of the gauge action (in units of the instanton action S_{Inst}) on the lattice, (blue) diamonds, with the gauge action of the continuum object, (red) line, given by Eq. (15) as function of the temporal extent Δt of the spherical vortex. The radius as well as the thickness of the vortex are given by $R = d = 6$. The midpoint of the vortex is half integer. b) Comparison of the gluonic topological charge $Q(\Delta t)$ for three values of $R = d$. The lattice results (calculated with the plaquette definition) are shown as (green) squares, (blue) diamonds and (black) circles. In the continuum, $Q(\Delta t) = 1$ for all Δt as shown by the (red) horizontal line. The lattice sizes $N_s \times N_t$ are chosen to fit the spacial and temporal extents of the spherical vortex.

zeromode is more or less located at the vortex. For the calculations, we used the overlap Dirac operator. With the described setting, we see that, as expected, the scalar density of the zeromode is smeared out in the temporal direction for the vortices with bigger temporal extent. The spatial localization, however, is more or less independent of the temporal extent of the vortex. Thus, all what really matters is that the vacuum to vacuum transition occurs, not how fast it occurs. Let us discuss this in a little bit more detail. First, we note that the scalar density of the zeromode is distributed, almost perfectly, spherically symmetric around the midpoint \vec{r}_0 of the vortex. Keeping this in mind, the discussion of the localization in 3 dimensional space reduces to a discussion of the localization in $|\vec{r} - \vec{r}_0|$. Thus we study

$$\rho(r, t) = \frac{1}{N(r)} \sum_{\vec{r}} \delta(r, |\vec{r} - \vec{r}_0|) \psi_0^\dagger(\vec{r}, t) \psi_0(\vec{r}, t) \quad \text{with} \quad N(r) = \sum_{\vec{r}} \delta(r, |\vec{r} - \vec{r}_0|). \quad (19)$$

In other words, the quantity $\rho(r, t)$ stands for the mean value of the scalar density of the zeromode, calculated for points belonging to the same $|\vec{r} - \vec{r}_0|$ and t . In Fig. 2 the results for $\rho(r, t)$ for generalized spherical vortices with $\Delta t = 1$ and $\Delta t = 5$ are compared. One can see the similarity between the result for the two different Δt .

VI. DISCUSSION AND OUTLOOK

In this paper, we identified the continuum object corresponding to the previously considered spherical vortex as a vacuum to vacuum transition in temporal direction. The discrepancy between the gluonic lattice topological charge and the index of the lattice Dirac operator, described in previous papers, turned out to be a discretization effect in the temporal direction. Starting from the continuum spherical vortex, we constructed a generalized spherical vortex on the lattice. We demonstrated the similarity to the original spherical vortex by using fermions as probes. We also showed that, with an appropriate choice of parameters, the action of the generalized spherical vortex can be quite small, as small as about 5/3 of the one-instanton action. For more details see [21].

It is known that topological charge contributions from center vortices, due to intersection (and writhing) points, emerge from vortex structures lying in a single $U(1)$ subgroup [22]. Color rotations in such structures are suppressed by the action. For example, intersections of vortices with orthogonal color structures give maximally negative plaquettes and are thus suppressed in the continuum limit. However, this does not mean that the whole vortex structure can be gauge transformed to a single $U(1)$ subgroup. We conjecture, for entropic reasons, that in the confined phase the color structure of center vortices contributes to the topological charge and chiral symmetry breaking.

Further investigations should therefore aim at quantifying these kinds of contributions in Monte Carlo generated configurations. First measurements show that the total vortex surface definitely covers the full S^3 , *i.e.* vortex plaquettes are distributed uniformly among the entire $SU(2)$ color palette. However, color vectors are not gauge

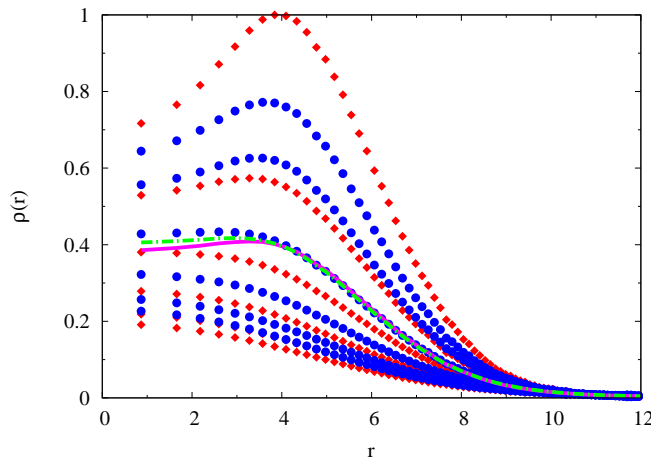


FIG. 2. $\rho(r, t)$ as defined in Eq. (19) with normalization $\rho_{\max} = 1$. The configuration investigated is given by Eq. (18) with $\Delta t = 1$, (red) diamonds, and $\Delta t = 5$, (blue) circles. The parameters of $\alpha(r)$ are given by $R = d = 6$ in both cases. The calculations have been performed on a $24^3 \times 12$ lattice. The different curves belong to different values of t . For $\Delta t = 1$ we have $t \in \{1 \triangleq 2, 3 \triangleq 12, \dots, 7 \triangleq 8\}$ and for $\Delta t = 5$ we have $t \in \{3 \triangleq 4, 5 \triangleq 2, \dots, 9 \triangleq 10\}$. The highest curve corresponds to the first pair of t -values, the lowest curve to the last pair of t -values. In between the curves fall monotonically. The solid (magenta) line represents the average over t for $\Delta t = 1$, the dash-dotted (green) line represents the same for $\Delta t = 5$.

invariant and therefore the number of full coverings is not so easy to measure, even though it is a gauge invariant, topological quantity. Further, writhing points dominate the topological susceptibility [23, 24], even in $SU(3)$ [25], and therefore color contributions might only play a sub-dominant role like intersection points. But the colorful spherical vortex may act as an ansatz for a model of chiral symmetry breaking, as it shows properties similar to instantons [26].

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